

Classical Polylogarithms

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1 Introduction

This article is an introduction to classical polylogarithms. After establishing some of their basic properties, we present several examples of Spencer Bloch where the dilogarithm is used to construct the second regulator. We also construct the polylogarithm local systems and show that each underlies a Tate variation of mixed Hodge structure [13]. We conclude by giving an exposition of a motivic description of the polylogarithm local systems. Most of the results in this paper were discovered by Bloch, Deligne, Ramakrishnan, Suslin and Beilinson.

Let k be a positive integer. The k th *polylogarithm* $\ln_k x$ is defined by

$$\ln_k x = \sum_{n=1}^{\infty} \frac{x^n}{n^k}. \quad (1)$$

This converges in the unit disk to a holomorphic function. The first polylogarithm, $\ln_1 x$, is just $-\log(1-x)$. The second,

$$\ln_2 x = \sum_{k=1}^{\infty} \frac{x^n}{n^2}$$

is called the *dilogarithm*, and was defined by Euler in 1768. The higher polylogarithms were defined by Spence in 1809 (cf. [26]).

It is believed that the k th regulator

$$c_k : K_m(X) \rightarrow H_{\mathcal{D}}^{2k-m}(X, \mathbb{Z}(k))$$

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from the algebraic K -theory of a complex algebraic variety X (and therefore all varieties of finite type over \mathbb{Q}) to its Deligne cohomology can be expressed in terms of the k th polylogarithm. If true, this would generalize the classical fact that the logarithm occurs as the first Chern class

$$c_1 : K_0(X) \rightarrow H^2(X, \mathbb{Z}(1))$$

and its single valued cousin, $\log | \cdot | : \mathbb{C}^* \rightarrow \mathbb{R}$, occurs as the regulator

$$c_1 : K_1(\mathbb{C}) \approx \mathbb{C}^* \rightarrow \mathbb{R} \approx H_{\mathcal{D}}^1(\text{spec } \mathbb{C}, \mathbb{R}(1)).$$

In the case where X is $\text{spec } \mathbb{C}$, this should mean that some single valued cousin $D_k : \mathbb{C} - \{0, 1\} \rightarrow \mathbb{R}$ of \ln_k should represent a multiple of the Borel regulator element

$$b_k \in H^{2k-1}(GL_k(\mathbb{C})^\delta, \mathbb{R}),$$

the cohomology class which gives rise to the regulator [6] (cf. also [35, §2.2], [16, §7.2]. (Here, $GL_k(\mathbb{C})^\delta$ denotes the general linear group viewed as a discrete group.) The cocycle condition would then be a functional equation satisfied by D_k which generalizes the 3-term functional equation satisfied by $D_1 = \log | \cdot |$. It is further believed that all of the rational K -theory of a field F should come from $F - \{0, 1\}$, and that the relations should all correspond to canonical functional equations satisfied by the D_k . Such statements are often referred to as the Zagier conjecture.² For example, for all fields, we may express the familiar fact

$$K_1(F) = F^\times$$

as

$$K_1(F) = \left[\coprod_{x \in F - \{0, 1\}} \mathbb{Z} \right] / \mathcal{R}$$

where the relations \mathcal{R} are generated by

$$[x] - [xy] + [y] = 0 \quad \text{and} \quad [x] + [x^{-1}] = 0,$$

²Zagier's original conjecture [42] asserted that the value at the positive integer m of the Dedekind zeta function of a number field F could be expressed as a determinant of values of D_m at F rational points of $\mathbb{P}^1 - \{0, 1, \infty\}$.

where $x, y \in F - \{0, 1\}$ and $xy \neq 1$. These are the analogues of the functional equations

$$D_1(x) - D_1(xy) + D_1(y) = 0 \quad \text{and} \quad D_1(x) + D_1(x^{-1}) = 0.$$

Note also that the functional equation in this case is precisely the condition that D_1 represent an element of $H^1(GL_1(\mathbb{C}), \mathbb{R})$. The corresponding story for the dilogarithm has been worked out by Bloch [5] and Suslin [37]. We give an account of this story in Section 4.

Useful references for basic material in this paper include [30] and [this volume] for algebraic K -theory, [26] for a comprehensive reference on classical aspects of polylogarithms, [25] for Tate variations of mixed Hodge structure, and [20] for basic facts about iterated integrals and the mixed Hodge theory of the fundamental group. The book [27] is a useful reference for more recent developments. (Also, references to other articles in this book.)

Notation: The group of units of a ring R will be denoted by R^\times . When Λ is \mathbb{Z} , \mathbb{Q} , or \mathbb{R} , $\Lambda(k)$ will denote the subgroup $(2\pi i)^k \Lambda$ of \mathbb{C} . It will also be used to denote the Hodge structure of type $(-k, -k)$ which has this abelian group as its lattice.

2 Monodromy

An easy power series manipulation yields the formula

$$\ln_k x = \int_0^x \ln_{k-1} z \frac{dz}{z},$$

where x lies in the unit disk and $k \geq 2$. It follows, by an induction argument, that each polylogarithm can be analytically continued to a multivalued holomorphic function on $\mathbb{C} - \{0, 1\}$. In this section we determine the monodromy of the polylogarithms. This was first computed by Ramakrishnan in [33].

Set

$$\omega_0 = \frac{dz}{z} \quad \text{and} \quad \omega_1 = \frac{dz}{1-z}.$$

Let

$$\omega = \begin{pmatrix} 0 & \omega_1 & 0 & \cdots & 0 \\ & \ddots & \omega_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & \omega_0 \\ 0 & \cdots & & & 0 \end{pmatrix} \in H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1(\log\{0, 1, \infty\})) \otimes gl_n(\mathbb{C}).$$

Consider the first order linear differential equation

$$d\lambda = \lambda\omega \tag{2}$$

where λ is a possibly multivalued function $\mathbb{C} - \{0, 1\} \rightarrow \mathbb{C}^{n+1}$.

Denote the k th power of the standard logarithm $\log x = \int_1^x \omega_0$ by $\log^k x$. Let

$$\Lambda(x) = \begin{pmatrix} 1 & \ln_1 x & \ln_2 x & \cdots & \cdots & \ln_n x \\ 0 & 2\pi i & 2\pi i \log x & & \cdots & \frac{2\pi i}{n!} \log^{n-1} x \\ 0 & 0 & (2\pi i)^2 & \ddots & \cdots & \frac{(2\pi i)^2}{(n-1)!} \log^{n-1} x \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & 0 & (2\pi i)^{n-1} & (2\pi i)^{n-1} \log x \\ 0 & \cdots & \cdots & \cdots & 0 & (2\pi i)^n \end{pmatrix} \in gl_{n+1}(\mathbb{C}).$$

More precisely,

$$\Lambda_{jk}(x) = \begin{cases} \ln_k x & \text{when } j = 0 \text{ and } k > 0; \\ \frac{(2\pi i)^j}{(k-j)!} \log^{k-j} x & \text{when } j, k > 0; \\ 0 & \text{when } k < j. \end{cases}$$

We will view this as a multivalued $gl_n(\mathbb{C})$ -valued function on $\mathbb{C} - \{0, 1\}$. By the *principal branch* of $\Lambda(x)$ we shall mean the matrix-valued function on the disk $|x - 1/2| < 1/2$ obtained by taking the standard branches of each of its entries on that disk. (The principal branch of \ln_k on this disk is the one given by the power series expansion (1).)

Proposition 2.1 *The function $\Lambda(x)$ is a fundamental solution of (2). That is,*

$$d\Lambda = \Lambda\omega$$

and $\Lambda(x)$ is non-singular for each $x \in \mathbb{C} - \{0, 1\}$. \square

If we analytically continue the principal branch of $\Lambda(x)$ about a loop in $\mathbb{C} - \{0, 1\}$ based at $1/2$, the resulting matrix of functions will still be a fundamental solution of (2). It follows that, for each loop γ based at $1/2$, there is a matrix $M(\gamma) \in GL_{n+1}(\mathbb{C})$ such that the analytic continuation of (the principal branch of) $\Lambda(x)$ about γ is $M(\gamma)\Lambda(x)$. For a pair of loops α, β based at $1/2$, we have

$$M(\alpha\beta) = M(\alpha)M(\beta).$$

Since the value of $M(\gamma)$ depends only on the homotopy class of γ , we obtain a monodromy representation

$$M : \pi_1(\mathbb{C} - \{0, 1\}, 1/2) \rightarrow GL_{n+1}(\mathbb{C}). \quad (3)$$

Let $\sigma_0, \sigma_1 \in \pi_1(\mathbb{C} - \{0, 1\}, 1/2)$ be the loops defined by

$$\sigma_0(t) = e^{2\pi it}/2, \quad \sigma_1(t) = 1 - e^{2\pi it}/2, \quad 0 \leq t \leq 1.$$

These loops generate $\pi_1(\mathbb{C} - \{0, 1\}, 1/2)$.

Proposition 2.2 [33] *We have*

$$M(\sigma_0) = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & J & \\ 0 & & & \end{array} \right) \text{ and } M(\sigma_1) = \left(\begin{array}{c|cccc} 1 & -1 & 0 & \cdots & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{array} \right)$$

where

$$J = \exp \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}$$

Proof. The monodromy around σ_0 is easy to calculate. Indeed, since the principal branch of each polylogarithm is single valued on the disk $|z - 1/2| \leq 1/2$, each is unchanged when continued along σ_0 . The formula for $M(\sigma_0)$ follows as the analytic continuation of $\log x$ about σ_0 is $\log x + 2\pi i$.

Since the principal branch of $\log x$ is defined in a neighbourhood of \mathbb{R}_+ , it follows that $\log x$ is invariant under analytic continuation along σ_1 .

We compute the analytic continuation of \ln_k along σ_1 by induction. When $k = 1$, $\ln_1 x$ changes to $\ln_1 x - 2\pi i$. Assume now that $k \geq 1$ and that the continuation of $\ln_k x$ along σ_1 is

$$\ln_k x - \frac{2\pi i}{(k-1)!} \log^{k-1} x.$$

Denote the integral of $f(z)dz$ along the straight line interval in the complex plane between the points a and b by

$$\int_a^b f(z)dz.$$

When $|x - 1/2| < 1/2$, we have

$$\begin{aligned} \ln_{k+1} x &= \int_0^x \ln_k z \frac{dz}{z} \\ &= \int_0^{1/2} \ln_k z \frac{dz}{z} + \int_{1/2}^x \ln_k z \frac{dz}{z}. \end{aligned} \quad (4)$$

The result of analytically continuing this along σ_1 is

$$\int_0^{1/2} \ln_k z \frac{dz}{z} + \int_{\sigma_1} \ln_k z \frac{dz}{z} + \int_{1/2}^x \ln_k z \frac{dz}{z}. \quad (5)$$

It follows from the inductive formula that the difference between (4) and (5) is

$$\int_{\sigma_1} \ln_k z \frac{dz}{z} - \frac{2\pi i}{(k-1)!} \int_{1/2}^x \log^{k-1} z \frac{dz}{z}. \quad (6)$$

For each $\epsilon \in (0, 1/2)$, the path which traverses the line segment from $1/2$ to $1 - \epsilon$, goes around the boundary of the disk $|z - 1| \leq \epsilon$ in the positive direction, then returns along the interval from $1 - \epsilon$ to $1/2$ represents the homotopy class σ_1 . Again, using the inductive formula for the monodromy of $\ln_k x$ around σ_1 , we have

$$\int_{\sigma_1} \ln_k z \frac{dz}{z} = -\frac{2\pi i}{(k-1)!} \int_{1-\epsilon}^{1/2} \log^{k-1} z \frac{dz}{z} + \int_{-\pi}^{\pi} \ln_k(1 + \epsilon e^{it}) \frac{d(\epsilon e^{it})}{1 + \epsilon e^{it}}.$$

The inductive hypothesis also implies that $\ln_k x$ is bounded in a neighbourhood of 1 when $k > 1$, so that the last integral $\rightarrow 0$ as $\epsilon \rightarrow 0$ for all k . (A separate argument is needed when $k = 1$.) Combining this with (6), we see that $\ln_{k+1} x$ changes by

$$-\frac{2\pi i}{(k-1)!} \left[\lim_{\epsilon \rightarrow 0} \int_{1-\epsilon}^{1/2} \log^{k-1} z \frac{dz}{z} + \int_{1/2}^x \log^{k-1} z \frac{dz}{z} \right] = -\frac{2\pi i}{k!} \log^k x$$

when continued around σ_1 . \square

The monodromy calculation has several interesting consequences. First, even though it does not make sense, in general, to talk about the value of a multivalued function at a point, it does make sense to talk about the value of \ln_k at 1.

Corollary 2.3 *The value of $(k-1)! \ln_k 1$ is well defined modulo $\mathbb{Z}(n)$ and is congruent to $(k-1)! \zeta(k)$. \square*

The second important consequence of the monodromy calculation is the rationality of the monodromy.

Corollary 2.4 *The image of the monodromy representation (3) is contained in $GL_{n+1}(\mathbb{Q})$. \square*

The significance of this last result is that it implies that the local system over $\mathbb{C} - \{0, 1\}$ which corresponds to the differential equation (2) is defined over \mathbb{Q} . This local system is called the *n th polylogarithm local system*. These local systems fit together to form an inverse system of local systems whose limit we call the *polylogarithm local system*. We now describe these local systems in detail.

Define a meromorphic connection ∇ on the trivial bundle

$$\mathbb{P}^1 \times \mathbb{C}^{n+1} \rightarrow \mathbb{P}^1 \tag{7}$$

by defining

$$\nabla f = df - f\omega$$

where $f : \mathbb{C} - \{0, 1\} \rightarrow \mathbb{C}^{n+1}$ is a section. This connection has regular singular points at 0, 1 and ∞ , and is flat over $\mathbb{C} - \{0, 1\}$ as ω satisfies the integrability condition

$$d\omega + \omega \wedge \omega = 0$$

(equivalently, because the equation $\nabla f = 0$ is a system of ordinary differential equations). Let $\lambda_0, \lambda_1, \dots, \lambda_n$ be the rows of $\Lambda(x)$. Each of these satisfies (2) and is therefore a flat section of (7). Even though these are multivalued, their \mathbb{Q} linear span is well defined as the monodromy representation is defined over \mathbb{Q} .

Suppose that X is a smooth curve and that \overline{X} is a smooth completion of X . Every flat bundle $E \rightarrow X$ has a *canonical extension* $\overline{E} \rightarrow \overline{X}$. Denote the local monodromy operator about a point $p \in D := \overline{X} - X$ by T_p . When each T_p is unipotent, the canonical extension is characterized by two properties. First, the meromorphic extension $\overline{\nabla}$ of the connection to $\overline{E} \rightarrow \overline{X}$ has logarithmic singularities along D . That is,

$$\overline{\nabla} : \overline{\mathcal{E}} \rightarrow \overline{\mathcal{E}} \otimes_{\mathcal{O}_{\overline{X}}} \Omega_{\overline{X}}^1(\log D).$$

Second, the residue of $\overline{\nabla}$ at each point of D is nilpotent.

Denote the \mathbb{Q} local system over $\mathbb{C} - \{0, 1\}$ which corresponds to the representation (3) by \mathbb{V} .

Since ω has nilpotent residue at 0, 1 and ∞ , we have:

Proposition 2.5 *The canonical extension of the flat holomorphic vector bundle $\mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_{\mathbb{C}-\{0,1\}}$ to \mathbb{P}^1 is the bundle (7) with the connection ∇ defined above. \square*

3 The Bloch-Wigner function

Define

$$D_2(x) = \Im \ln_2 x + \log |x| \arg(1 - x)$$

when $|x - 1/2| < 1/2$ and where $\ln_2 x$, $\log x$, and $\arg(1 - x)$ denote the principal branches of these functions in the disk $|x - 1/2| < 1/2$. An easy computation using the monodromy calculation (2.2) shows that D_2 is invariant under continuation along the generators σ_0 and σ_1 of $\pi_1(\mathbb{C} - \{0, 1\}, 1/2)$.

Consequently, the function D_2 extends to a single valued, real analytic function

$$D_2 : \mathbb{C} - \{0, 1\} \rightarrow \mathbb{R}.$$

This is called the *Bloch-Wigner function*. If we define

$$D_2(0) = D_2(1) = D_2(\infty) = 0$$

then D_2 is a continuous function $D_2 : \mathbb{P}^1 \rightarrow \mathbb{R}$. The Bloch-Wigner function should be viewed as having the same relation to \ln_2 as $D_1 := \log | \cdot |$ bears to the logarithm.

The boundary of hyperbolic 3-space \mathbb{H}^3 is the Riemann sphere \mathbb{P}^1 . The group of orientation preserving isometries of \mathbb{H}^3 is $PSL_2(\mathbb{C})$. The induced action on the boundary is just the standard action of $PSL_2(\mathbb{C})$ on \mathbb{P}^1 via fractional linear transformations.

Denote the ideal tetrahedron in \mathbb{H}^3 with vertices at $a_0, a_1, a_2, a_3 \in \mathbb{P}^1$ by $\Delta(a_0, a_1, a_2, a_3)$. Since the volume form of hyperbolic space is invariant under the action of the isometry group,

$$\text{vol } \Delta(a_0, a_1, a_2, a_3) = \text{vol } \Delta(\lambda, 1, 0, \infty).$$

where λ is the cross ratio $[a_0 : a_1 : a_2 : a_3]$ of the vertices.

The following result goes back to Lobachevsky (cf. [30]). A proof may be found in [17, p. 172].

Theorem 3.1 *For each $z \in \mathbb{P}^1$, the volume of $\Delta(z, 1, 0, \infty)$ equals $D_2(z)$.*

Corollary 3.2 *If $a_0, a_1, a_2, a_3, a_4 \in \mathbb{P}^1$, then*

$$\sum_{j=0}^4 (-1)^j D_2([a_0 : \cdots : \widehat{a_j} : \cdots : a_4]) = 0.$$

Moreover, for all permutations σ of $\{0, 1, 2, 3\}$,

$$D_2([a_{\sigma(0)} : \cdots : a_{\sigma(3)}]) = \text{sgn}(\sigma) D_2([a_0 : \cdots : a_3]).$$

Proof. The first assertion follows as the ideal polyhedron P with vertices a_0, a_1, a_2, a_3, a_4 decomposes into a union of ideal tetrahedra in 2 different ways. Viz.

$$P = \Delta(a_1, a_2, a_3, a_4) \cup \Delta(a_0, a_1, a_3, a_4) \cup \Delta(a_0, a_1, a_2, a_3)$$

and

$$P = \Delta(a_0, a_2, a_3, a_4) \cup \Delta(a_0, a_1, a_2, a_4)$$

In each case the pieces intersect along 2-dimensional faces. The first assertion follows from Theorem 3.1 by comparing volumes. The second follows as swapping the order of 2 vertices reverses the orientation of the tetrahedron.

□

Taking the five points to be $y, x, 1, 0, \infty$, we obtain the usual form of the functional equation, which is the analogue for D_2 of the Abel-Spence functional equation of \ln_2 .

Corollary 3.3 *If $y, x, 1, 0, \infty$ are distinct points of \mathbb{P}^1 , then*

$$D_2(x) - D_2(y) + D_2(y/x) - D_2((1-y)/(1-x)) + D_2((1-y^{-1})/(1-x^{-1})) = 0. \quad \square$$

Ramakrishnan [34] showed that all the polylogarithms have such single valued cousins. The essential point being that one can use the unipotence of the monodromy group and induction to kill off the monodromy of \ln_k . Zagier [42] gave an explicit formula for Ramakrishnan's functions. In general, there seems to be no canonical way to go from the multivalued polylogarithm to a single valued function. Goncharov [18] has modified Ramakrishnan's trilogarithm and proved that his function satisfies a very natural functional equation. His function is defined by

$$D_3(x) = \Re \left[\ln_3(x) - \log |x| \ln_2 x + (\log^2 |x| \ln_1 x)/3 \right]. \quad (8)$$

It is single valued and continuous on \mathbb{P}^1 , and real analytic on $\mathbb{P}^1 - \{0, 1, \infty\}$.

4 The regulator $K_3(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{Q}(2)$

The Deligne cohomology of $\text{spec } \mathbb{C}$ is

$$H_{\mathcal{D}}^m(\text{spec } \mathbb{C}, \Lambda(k)) = \begin{cases} 0 & m \neq 1; \\ \mathbb{C}/\Lambda(k) & m = 1, \end{cases}$$

where Λ denotes \mathbb{Z} , \mathbb{Q} or \mathbb{R} . So the only non-trivial regulators for $\text{spec } \mathbb{C}$ with values in Deligne cohomology are

$$K_{2m-1}(\mathbb{C}) \rightarrow H_{\mathcal{D}}^1(\text{spec } \mathbb{C}, \Lambda(m)) \approx \mathbb{C}/\Lambda(m)$$

for each $m \geq 1$. The first regulators

$$K_1(\mathbb{C}) \approx \mathbb{C}^* \rightarrow \mathbb{C}/\mathbb{Z}(1) \quad \text{and} \quad K_1(\mathbb{C}) \approx \mathbb{C}^* \rightarrow \mathbb{C}/\mathbb{R}(1) \approx \mathbb{R}$$

are given by the functions \log and $\log |\cdot|$, respectively. In this section we give an account of the construction of the second regulators

$$K_3(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{Q}(2) \quad \text{and} \quad K_3(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{R}(2) \approx \mathbb{R}(1)$$

using \ln_2 and D_2 respectively. These results go back to Bloch and Wigner (unpublished, cf. [17]). As is customary, $GL_n(\mathbb{C})^\delta$ signifies that $GL_n(\mathbb{C})$ is viewed as a group with the discrete topology.

Lemma 4.1 *The Bloch-Wigner function defines a canonical group cohomology class*

$$\mathcal{D}_2 \in H^3(GL_2(\mathbb{C})^\delta, \mathbb{R}).$$

Proof. Let F be a field and $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, define $C_k(F, n)$ to be the free abelian group with basis ordered $(k+1)$ -tuples (v_0, \dots, v_k) , of vectors v_j of F^n in general position. (I.e., each $\min(n, k+1)$ of them are independent.) Define a differential $\partial : C_k \rightarrow C_{k-1}$ by

$$\partial : (v_0, \dots, v_k) \mapsto \sum_{j=0}^k (-1)^j (v_0, \dots, \widehat{v_j}, \dots, v_k).$$

When F is infinite, the complex $C_\bullet(F, n)$ is quasi-isomorphic to \mathbb{Z} . Since $GL_n(F)$ acts on this complex, it is a resolution of the trivial module. So, for all $GL_n(F)$ modules M , there is a natural map

$$H^\bullet \left(\text{Hom}_{GL_n(F)}(C_\bullet(F, n), M) \right) \rightarrow H^\bullet(GL_n(F), M),$$

provided that F is infinite.

Denote the point in $\mathbb{P}^{n-1}(F)$ determined by the non-zero vector v of F^n by $[v]$.

Define a map $f : C_3(\mathbb{C}, 2) \rightarrow \mathbb{R}$ by

$$f(v_0, v_1, v_2, v_3) = D_2([v_0] : [v_1] : [v_2] : [v_3]).$$

The cocycle condition is simply the functional equation (3.2). Denote the image of this cohomology class in $H^3(GL_2(\mathbb{C})^\delta, \mathbb{R})$ by \mathcal{D}_2 . \square

A cohomology class $c \in H^m(GL_m(R), \Lambda)$ defines a map

$$K_m(R) \rightarrow \Lambda.$$

This map is obtained as the composite

$$K_m(R) = \pi_m(BGL(R)^+) \rightarrow H_m(BGL(R)^+) \approx H_m(GL(R)) \xrightarrow{c} \Lambda$$

of the Hurewicz homomorphism with the Λ -valued functional on homology induced by c .

Theorem 4.2 *The map*

$$K_3(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{R}(2)$$

defined by $i\mathcal{D}_2$ equals the Beilinson regulator, and this map equals half of the Borel regulator.

Proof. The proof is an assemblage of results from the literature, and we only give a sketch. First, the cocycle

$$GL_2(\mathbb{C})^4 \rightarrow \mathbb{C}/\mathbb{R}(2)$$

induced by iD_2 , composed with the cross ratio, represents a continuous cohomology class $i\mathcal{D}_2$, even though this cocycle is not itself continuous. This apparently contradictory state of affairs arises because the cross ratio $(\mathbb{P}^1)^4 \rightarrow \mathbb{P}^1$ is not everywhere defined. One can show (e.g. [40]) that the image of this class under the natural map

$$H_{\text{cts}}^3(GL_2(\mathbb{C}), \mathbb{C}/\mathbb{R}(2)) \rightarrow H^3(GL_2(\mathbb{C})^\delta, \mathbb{C}/\mathbb{R}(2))$$

is the class $i\mathcal{D}_2$ of (4.1).

Next, since

$$H_{\text{cts}}^3(GL_2(\mathbb{C}), \mathbb{C}/\mathbb{R}(2)) \approx H_{\text{cts}}^3(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{R}(2)) \approx \mathbb{C}/\mathbb{R}(2)$$

for all $n \geq 2$, and since this group is spanned by the continuous cohomology class β_2 used to define the Borel regulator (cf. [35], [16]), it follows that there is a real number λ such that

$$i\mathfrak{D}_2 = \lambda\beta_2/2.$$

Denote by c_k the k th Beilinson Chern class of the universal flat bundle over $BGL_n(\mathbb{C})^\delta$. This is an element of

$$H_{\mathcal{D}}^{2k}(BGL_n(\mathbb{C})^\delta, \mathbb{R}(k)) \approx H^{2k-1}(GL_n(\mathbb{C}), \mathbb{C}/\mathbb{R}(k)).$$

In [16] it is shown that, for all n and k , the classes c_k and $\beta_k/2$ in

$$H^k(GL_n(\mathbb{C})^\delta, \mathbb{C}/\mathbb{R}(2))$$

are equal. It follows that $i\mathfrak{D}_2$ is λ times the Beilinson Chern class

$$c_2 : K_3(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{R}(2).$$

Denote the rational K -theory $K_\bullet(R) \otimes \mathbb{Q}$ of a ring R by $K_\bullet(R)_\mathbb{Q}$. Since $BGL(R)^+$ is an H-space, the Hurewicz homomorphism

$$K_m(R)_\mathbb{Q} := \pi_m(BGL(R)^+) \otimes \mathbb{Q} \rightarrow H_m(BGL(R)^+, \mathbb{Q}) \approx H_m(GL(R), \mathbb{Q})$$

is injective. Define the *rank filtration*

$$r_1 K_m(R) \subseteq r_2 K_m(R) \subseteq r_3 K_m(R) \subseteq \cdots \subseteq K_m(R)_\mathbb{Q}$$

of $K_m(R)_\mathbb{Q}$ by

$$r_k K_m(R) = K_m(R)_\mathbb{Q} \cap \text{im} \{H_m(GL_k(R), \mathbb{Q}) \rightarrow H_m(GL(R), \mathbb{Q})\}.$$

Suslin [37] has proved that for all infinite fields F

$$K_m(F)_\mathbb{Q} = r_m K_m(F)$$

and that

$$K_m(F)_\mathbb{Q}/r_{m-1} K_m(F) \approx K_m^M(F) \otimes \mathbb{Q},$$

where $K_\bullet^M(F)$ denotes the Milnor K -theory of F .

In particular, for all infinite fields F , there is a canonical isomorphism

$$K_3(F)_{\mathbb{Q}} = K_3^M(F)_{\mathbb{Q}} \oplus r_2 K_3(F).$$

Since all elements of K_3^M of any field are decomposable, the generalization of the Whitney sum formula to K -theory implies that

$$c_2 : K_3(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{R}(2)$$

vanishes on $K_3^M(\mathbb{C})$. It follows that c_2 factors through the projection onto $r_2 K_3(\mathbb{C})$.

$$\begin{array}{ccc} K_3(\mathbb{C}) & \xrightarrow{c_2} & \mathbb{C}/\mathbb{R}(2) \\ \text{proj} \downarrow & \nearrow & \\ r_2 K_3(\mathbb{C}) & & \end{array}$$

Since $r_2 K_3(\mathbb{C})$ comes from $H_3(GL_2(\mathbb{C}))$, the restriction of c_2 to $r_2 K_3(\mathbb{C})$ is given by the class $i\lambda^{-1}\mathfrak{D}_2$. Now Dupont [15] has proved that the $i\mathfrak{D}_2$ equals the Cheeger-Simons class

$$\hat{c}_2 \in H^3(GL_2(\mathbb{C})^\delta, \mathbb{C}/\mathbb{R}(2)).$$

of the universal rank 2 flat bundle. But by the main result of [16], this class equals the Beilinson Chern class of the universal flat bundle. It follows that $\lambda = 1$. \square

Most of this story has been extended to the trilogarithm by Goncharov [18] and Yang [40]. In [23] a canonical single valued trilogarithm

$$S_3 : \left\{ \begin{array}{l} \text{ordered 6-tuples of points} \\ \text{in } \mathbb{P}^2, \text{ no 3 on a line} \end{array} \right\} / \text{projective equivalence} \rightarrow \mathbb{R}$$

was constructed. It satisfies the seven term functional equation

$$\sum_{j=0}^6 (-1)^j S_3(a_0, \dots, \widehat{a_j}, \dots, a_6) = 0$$

where a_0, \dots, a_6 are points in \mathbb{P}^2 , no 3 of which lie on a line. This equation is an obvious generalization of the 5-term equation satisfied by D_2 . As in (4.1), this function determines a class in $H^5(GL_3(\mathbb{C}), \mathbb{R})$.

As in the case of the dilogarithm, the cocycle condition is precisely the functional equation. Goncharov and Yang both showed that this class in

$H^5(GL_3(\mathbb{C}), \mathbb{R})$ is a non-zero rational multiple of the class which corresponds to the Beilinson Chern class of the universal flat bundle over $BGL_3(\mathbb{C})^\delta$. Goncharov found an explicit formula for S_3 in terms of his single valued version of the classical trilogarithm (8). The appropriate analogue of Suslin's theorem is not known for $K_5(\mathbb{C})$. However, Yang [41] proved the rank conjecture for all K -groups of all number fields except \mathbb{Q} . Their work enables one to find formulas for the regulator mapping

$$c_3 : K_5(F) \rightarrow [\mathbb{C}/\mathbb{R}(3)]^{r_1+r_2}$$

for all number fields F in terms of values of the trilogarithm of [23] in the case of Yang, and the classical trilogarithm in the case of Goncharov.

The regulator $c_2 : K_3(\mathbb{C}) \rightarrow \mathbb{C}/\mathbb{Q}(2)$ can be written in terms of the multivalued dilogarithm. This work goes back to unpublished work of Bloch and Wigner (cf. [17]). The construction is similar to, but more complicated than, the construction of the regulator given above. We only sketch the construction. More details can be found in [17].

For x in the disk $|x - 1/2| < 1/2$, define

$$\rho(x) = \frac{1}{2} \left[\log x \wedge \log(1-x) + 2\pi i \wedge \frac{1}{2\pi i} \left(\ln_2(1-x) - \ln_2(x) - \pi^2/6 \right) \right] \in \Lambda_{\mathbb{Z}}^2 \mathbb{C}.$$

Here all functions are taken to be the principal branches. This function is single valued, and therefore extends to a single valued function

$$\rho : \mathbb{C} - \{0, 1\} \rightarrow \Lambda_{\mathbb{Z}}^2 \mathbb{C}.$$

It satisfies a generalization of the 5-term equation satisfied by D_2 . If $x, y \in \mathbb{C} - \{0, 1\}$ and $x \neq y$, then

$$\rho(x) - \rho(y) + \rho(y/x) - \rho((1-y)/(1-x)) + \rho((1-y^{-1})/(1-x^{-1})) = 0.$$

Define $\mathcal{P}(F)$ to be the free abelian group generated by $F - \{0, 1\}$ subject to the relations

$$[x] - [y] + [y/x] - [(1-y)/(1-x)] + [(1-y^{-1})/(1-x^{-1})] = 0.$$

This is often called the *scissors congruence group*, or the *Bloch group*. The function ρ induces a map

$$\rho : \mathcal{P}(\mathbb{C}) \rightarrow \Lambda_{\mathbb{Z}}^2 \mathbb{C}.$$

There is a natural homomorphism

$$H_3(SL_2(F)) \rightarrow \mathcal{P}(F)$$

whose construction is similar to that of the homomorphism $H_3(GL_2(F)) \rightarrow H_\bullet(C_\bullet(F, n))$ given in the proof of (4.1). If F is algebraically closed of characteristic 0, then there is an exact sequence

$$0 \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow H_3(SL_2(F)) \rightarrow \mathcal{P}(F) \rightarrow \Lambda_{\mathbb{Z}}^2 F^\times \rightarrow K_2(F) \rightarrow 0.$$

A proof can be found in the appendix of [17]. The map $\mathcal{P}(F) \rightarrow \Lambda_{\mathbb{Z}}^2 F^\times$ takes the generator $[x]$ of $\mathcal{P}(F)$ to $(1 - x) \wedge x$. The most right hand map takes $x \wedge y$ to $\{x, y\}$.

The kernel of the map

$$\Lambda^2 \exp : \Lambda_{\mathbb{Z}}^2 \mathbb{C} \rightarrow \Lambda_{\mathbb{Z}}^2 \mathbb{C}^*$$

is $\mathbb{C}/\mathbb{Q}(2)$, where this is included in $\Lambda_{\mathbb{Z}}^2 \mathbb{C}$ by taking the coset of λ to $2\pi i \wedge (\lambda/2\pi i)$. Since the diagram

$$\begin{array}{ccc} \mathcal{P}(\mathbb{C}) & \rightarrow & \Lambda_{\mathbb{Z}}^2 \mathbb{C}^\times \\ \rho \downarrow & & \downarrow id \\ \Lambda_{\mathbb{Z}}^2 \mathbb{C} & \xrightarrow{\Lambda^2 \exp} & \Lambda_{\mathbb{Z}}^2 \mathbb{C}^* \end{array}$$

commutes, ρ induces a homomorphism

$$H_3(SL_2(\mathbb{C})) \rightarrow \mathbb{C}/\mathbb{Q}(2).$$

By [15], this represents the second Cheeger-Simons class c_2 of the universal flat bundle over $BSL_2(\mathbb{C})^\delta$. By the main result of [16], this equals the Beilinson Chern class of the universal flat bundle over $BSL_2(\mathbb{C})^\delta$. As above, c_2 vanishes on $K_3^M(\mathbb{C})$ and the diagram

$$\begin{array}{ccc} K_3(\mathbb{C}) & \xrightarrow{c_2} & \mathbb{C}/\mathbb{Q}(2) \\ \text{proj} \downarrow & \nearrow \rho & \\ r_2 K_3(\mathbb{C}) & & \end{array}$$

commutes. One can show that the map

$$H_3(SL_2(F), \mathbb{Q}) \rightarrow r_2 K_3(F)$$

is surjective. It follows that the map constructed above induces the regulator on all of $K_3(\mathbb{C})$. Alternatively, one can appeal to the theorem of Suslin [38] which asserts that there are natural isomorphisms

$$H_3(SL_2(F), \mathbb{Q}) \approx r_2 K_3(F) \approx \ker \left\{ \mathcal{P}(F) \rightarrow \Lambda_{\mathbb{Z}}^2 F^\times \right\} \otimes \mathbb{Q}$$

for all fields F . This last isomorphism says that all of the weight 2 part of K_3 of a field comes from $\mathbb{P}^1 - \{0, 1, \infty\}$ and that all the relations come from the functional equation of the dilogarithm. This generalizes the fact mentioned in the introduction that the relations in K_1 come from the functional equation of the logarithm. The analogue of this statement for the weight 3 part of K_5 is not known at this time, although Goncharov [18] has made significant progress.

5 Iterated integrals

At this stage it is convenient to introduce Chen's iterated integrals [9] Basic references for this section are [9], [20].

Suppose that M is a manifold and that $w_1 \dots w_r$ are smooth \mathbb{C} -valued 1-forms on M . For each piecewise smooth path $\gamma : [0, 1] \rightarrow M$, we can define

$$\int_{\gamma} w_1 \dots w_r = \int \dots \int_{0 \leq t_1 \leq \dots \leq t_r \leq 1} f_1(t_1) \dots f_r(t_r) dt_1 \dots dt_r,$$

where $\gamma^* w_j = f_j(t) dt$ for each j . This can be viewed as a \mathbb{C} -valued function

$$\int_{\gamma} w_1 \dots w_r : PM \rightarrow \mathbb{C}$$

on the the space of piecewise smooth paths in M . When $r = 1$, $\int_{\gamma} w$ is just the usual line integral. An *iterated integral* is any function $PM \rightarrow \mathbb{C}$ which is a linear combination of constant functions and basic iterated integrals

$$\int_{\gamma} w_1 \dots w_r.$$

Now let $M = \mathbb{C} - \{0, 1\}$ and

$$\omega_0 = \frac{dz}{z} \text{ and } \omega_1 = \frac{dz}{1-z}.$$

Then

$$\ln_1 x = -\log(1-x) = \int_0^x \omega_1.$$

By induction and the definition, we have, for all $k \geq 2$,

$$\ln_k x = \int_0^x \ln_{k-1} z \omega_0 = \int_0^x \omega_1 \overbrace{\omega_0 \dots \omega_0}^{k-1 \text{ times}}.$$

Here the path of integration must be chosen so that once it has left 0, it never passes through 0 or 1 on its way to x .

The basic properties of iterated integrals are summarized in the following proposition.

Proposition 5.1 [9, 20] *Suppose that w_1, w_2, \dots are \mathbb{C} -valued 1-forms on a manifold M .*

- (i) *The value of $\int_\gamma w_1 \dots w_r$ is independent of the parameterization of γ .*
- (ii) *If $\alpha, \beta : [0, 1] \rightarrow M$ are composable paths (i.e. $\alpha(1) = \beta(0)$), then*

$$\int_{\alpha\beta} w_1 \dots w_r = \sum_{j=0}^r \int_\alpha w_1 \dots w_j \int_\beta w_{j+1} \dots w_r.$$

Here, $\int_\gamma \phi_1 \dots \phi_m$ is to be interpreted as 1 when $m = 0$.

- (iii) *For all paths γ ,*

$$\int_{\gamma^{-1}} w_1 \dots w_r = (-1)^r \int_\gamma w_r \dots w_1.$$

- (iv) *For all paths α in M ,*

$$\int_\alpha w_1 \dots w_r \int_\alpha w_{r+1} \dots w_{r+s} = \sum_\sigma \int_\alpha w_{\sigma(1)} \dots w_{\sigma(r+s)},$$

where σ ranges over all shuffles of type (r, s) .

6 The Regulator $K_2(X) \rightarrow H^1(X, \mathbb{C}^*)$

This section is an exposition of Bloch's construction of the regulator

$$c_2 : K_2(X) \rightarrow H_{\mathcal{D}}^2(X, \mathbb{Z}(2))$$

using the dilogarithm. We have freely incorporated the elegant approaches of Deligne [13] and Ramakrishnan [32, 35].

When X is a curve, there is a natural isomorphism

$$H^2(X, \mathbb{Z}(2)) \approx H^1(X, \mathbb{C}/\mathbb{Z}(2)).$$

Identifying $\mathbb{C}/\mathbb{Z}(2)$ with \mathbb{C}^* by the map $\lambda \mapsto \exp[\lambda/2\pi i]$, we obtain a canonical identification of $H_{\mathcal{D}}^2(X, \mathbb{Z}(2))$ with $H^1(X, \mathbb{C}^*)$, the group of flat line bundles over X .

We set

$$H_{\mathbb{Z}} = \begin{pmatrix} 1 & \mathbb{Z}(1) & \mathbb{Z}(2) \\ 0 & 1 & \mathbb{Z}(1) \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$H_{\mathbb{C}} = \begin{pmatrix} 1 & \mathbb{C} & \mathbb{C} \\ 0 & 1 & \mathbb{C} \\ 0 & 0 & 1 \end{pmatrix}$$

There is a natural bundle projection

$$H_{\mathbb{Z}} \backslash H_{\mathbb{C}} \rightarrow \mathbb{C}^* \times \mathbb{C}^*; \tag{9}$$

it takes the coset of

$$\begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}$$

to (e^u, e^v) . It has fiber $\mathbb{C}/\mathbb{Z}(2)$, which we identify with \mathbb{C}^* as above.

Proposition 6.1 *There is a natural connection on this bundle with curvature $(dx/x) \wedge (dy/y)/2\pi i$, where x and y are the coordinates in $\mathbb{C}^* \times \mathbb{C}^*$.*

Proof. First consider the pullback of the bundle (9) to $\mathbb{C} \times \mathbb{C}$

$$\mathbb{Z}(2) \backslash H_{\mathbb{C}} \rightarrow \mathbb{C} \times \mathbb{C} \quad (10)$$

along the map $(u, v) \mapsto (e^u, e^v)$. The map

$$H_{\mathbb{C}} \rightarrow \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$$

defined by

$$\begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \mapsto (\exp(w/2\pi i), u, v)$$

induces an isomorphism of $\mathbb{Z}(2) \backslash H_{\mathbb{C}}$ with $\mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$ which commutes with the projections to $\mathbb{C} \times \mathbb{C}$. So the bundle (10) is trivial, and sections of it can be identified with maps $\zeta : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}^*$. Define a connection on this bundle by

$$\nabla \zeta = d\zeta - \zeta u dv / 2\pi i$$

This connection is easily seen to be invariant under the left action

$$(n, m) : (\zeta, u, v) \mapsto (e^{nv} \zeta, u + 2\pi i n, v + 2\pi i m)$$

of $\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$ induced by the left action of $H_{\mathbb{Z}}$ on $H_{\mathbb{C}}$. It therefore descends to a connection on the bundle (9). The connection form of the pullback bundle is $u dv / 2\pi i$, from which it follows that its curvature is $du \wedge dv / 2\pi i$ and that the curvature of (9) is $(dx/x) \wedge (dy/y) / 2\pi i$. \square

Now suppose that X is a smooth curve over \mathbb{C} . Denote the function field of X by $\mathbb{C}(X)$ and the generic point $\text{spec } \mathbb{C}(X)$ of X by η_X . We first define the regulator on $K_2(\eta_X) := K_2(\mathbb{C}(X))$. The Deligne cohomology of η_X is defined by

$$H_{\mathcal{D}}^m(\eta_X, \Lambda(k)) = \varinjlim H_{\mathcal{D}}^m(U, \Lambda(k))$$

where the limit is taken over all Zariski open subsets U of X . In particular, we have

$$H_{\mathcal{D}}^2(\eta_X, \mathbb{Z}(2)) = \varinjlim H^1(U, \mathbb{C}^*).$$

This latter group is the group of flat line bundles at the generic point of X —elements of this group are flat line bundles defined on some Zariski open subset of X , and two such are identified if they agree on a smaller open subset. The product is tensor product.

By Matsumoto's Theorem [29], $K_2(\eta_X)$ is generated by symbols $\{f, g\}$, where $f, g \in \mathbb{C}(X)^\times$. The only relations which hold between these symbols are bilinearity

$$\{f_1 f_2, g\} = \{f_1, g\} \{f_2, g\}, \quad \{f, g_1 g_2\} = \{f, g_1\} \{f, g_2\}$$

and the Steinberg relation

$$\{1 - f, f\} = 1$$

whenever f and $1 - f$ are both in $\mathbb{C}(X)^\times$.

Now suppose that $\{f, g\} \in K_2(\eta_X)$. There is a Zariski open subset U of X such that f and g are both defined and invertible on U . They therefore define a regular function

$$(f, g) : U \rightarrow \mathbb{C}^* \times \mathbb{C}^*.$$

The pullback of the line bundle $H_{\mathbb{Z}} \backslash H_{\mathbb{C}}$ to U is flat as it has curvature a multiple of $(df/f) \wedge (dg/g)$, which is zero as U is a curve. Denote it by $\langle f, g \rangle$. It is an element of $H^1(U, \mathbb{C}^*)$, and therefore of $H^1(\eta_X, \mathbb{C}^*)$.

Remark 6.2 Observe that this construction makes sense when X is a Riemann surface and $\mathbb{C}(X)$ denotes the field of meromorphic functions of X . It is natural with respect to holomorphic maps between Riemann surfaces.

Proposition 6.3 *If f, g are invertible functions on the Zariski open subset U of X , the monodromy of the flat bundle $\langle f, g \rangle$ about a loop γ based at $p \in U$ is*

$$I(f, g, \gamma) := \int_{\gamma} \frac{df}{f} \frac{dg}{g} - \log g(p) \int_{\gamma} \frac{df}{f} + \log f(p) \int_{\gamma} \frac{dg}{g} \in \mathbb{C}/\mathbb{Z}(2).$$

Proof. We will deduce the result by computing the monodromy about a loop γ in $\mathbb{C}^* \times \mathbb{C}^*$. The assertion will then follow by pulling back the answer to U . Let γ be a path in $\mathbb{C}^* \times \mathbb{C}^*$ which begins at (x_0, y_0) . For $t \in [0, 1]$, denote the path $s \mapsto \gamma(st)$ by γ_t . The horizontal lift of γ to $H_{\mathbb{Z}} \backslash H_{\mathbb{C}}$ which begins at the coset of

$$\begin{pmatrix} 1 & \log x_0 & w \\ 0 & 1 & \log y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

is

$$t \mapsto H_{\mathbb{Z}} \begin{pmatrix} 1 & \log x_0 & w \\ 0 & 1 & \log y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \int_{\gamma_t} \frac{dx}{x} & \int_{\gamma_t} \frac{dx}{x} \frac{dy}{y} \\ 0 & 1 & \int_{\gamma_t} \frac{dy}{y} \\ 0 & 0 & 1 \end{pmatrix}.$$

That is,

$$t \mapsto \begin{pmatrix} 1 & \log x_0 + \int_{\gamma_t} \frac{dx}{x} & w + \int_{\gamma_t} \frac{dx}{x} \frac{dy}{y} + \log x_0 \int_{\gamma_t} \frac{dy}{y} \\ 0 & 1 & \log y_0 + \int_{\gamma_t} \frac{dy}{y} \\ 0 & 0 & 1 \end{pmatrix}$$

Now suppose that γ is a loop. Then the endpoint of the horizontal lift of γ is congruent to the matrix

$$\begin{aligned} & \begin{pmatrix} 1 & -\int_{\gamma} \frac{dx}{x} & \int_{\gamma} \frac{dx}{x} \int_{\gamma} \frac{dy}{y} \\ 0 & 1 & -\int_{\gamma} \frac{dy}{y} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \log x_0 + \int_{\gamma} \frac{dx}{x} & w + \int_{\gamma} \frac{dx}{x} \frac{dy}{y} + \log x_0 \int_{\gamma} \frac{dy}{y} \\ 0 & 1 & \log y_0 + \int_{\gamma} \frac{dy}{y} \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \log x_0 & w + \int_{\gamma} \frac{dx}{x} \frac{dy}{y} + \log x_0 \int_{\gamma} \frac{dy}{y} - \log y_0 \int_{\gamma} \frac{dx}{x} \\ 0 & 1 & \log y_0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

modulo the left action of $H_{\mathbb{Z}}$. It follows that the holonomy about γ is

$$\int_{\gamma} \frac{dx}{x} \frac{dy}{y} + \log x_0 \int_{\gamma} \frac{dy}{y} - \log y_0 \int_{\gamma} \frac{dx}{x} \pmod{\mathbb{Z}(2)}.$$

The result follows by pulling the result back to U along the map (f, g) . \square

Proposition 6.4 *If $f, f_1, f_2, g \in \mathbb{C}(X)^{\times}$, then*

$$\langle f_1 f_2, g \rangle = \langle f_1, g \rangle \langle f_2, g \rangle \quad \text{and} \quad \langle f, g \rangle^{-1} = \langle g, f \rangle.$$

Moreover, if $f, 1 - f \in \mathbb{C}(X)^$, then*

$$\langle 1 - f, f \rangle = 1.$$

Proof. It is clear from properties of the logarithm and (5.1) that

$$I(f_1 f_2, g, \gamma) = I(f_1, g, \gamma) + I(f_2, g, \gamma).$$

It is not difficult to use (5.1) to prove that

$$I(f, g, \gamma) + I(g, f, \gamma) = 0.$$

These imply the linearity and skew symmetry of the symbol $\langle f, g \rangle$.

It suffices to prove the Steinberg relation in the universal case where $U = \mathbb{C} - \{0, 1\}$ and $f = x$. The line bundle $\langle 1 - x, x \rangle$ is trivial as a flat bundle if and only if it has a flat section. The dilogarithm provides such a section. Define $s : \mathbb{C} - \{0, 1\} \rightarrow H_{\mathbb{Z}} \backslash H_{\mathbb{C}}$ by

$$s(x) = H_{\mathbb{Z}} \begin{pmatrix} 1 & \log(1 - x) & -\ln_2 x \\ 0 & 1 & \log x \\ 0 & 0 & 1 \end{pmatrix}$$

This section is flat, so the Steinberg relation holds. \square

Theorem 6.5 [2] *Taking $\{f, g\}$ to $\langle f, g \rangle$ defines a map*

$$K_2(\eta_X) \rightarrow H^1(\eta_X, \mathbb{C}^*)$$

which is the Chern class c_2 .

Proof. The first assertion is an easy consequence of Matsumoto's description of K_2 and (6.4). The second follows from the fact that the symbol $\{f, g\}$ is the cup product of $f, g \in K_1(\eta_X) \approx \mathbb{C}(X)^\times$. Properties of Chern classes then imply that

$$c_2(\{f, g\}) = c_1(f) \cup c_1(g)$$

where the right hand side is the cup product of

$$c_1(f), c_1(g) \in H_{\mathcal{D}}^2(\eta_X, \mathbb{Z}(1)) \approx \mathbb{C}(X)^*.$$

Under this isomorphism, c_1 is just the identity.

The formula for the cup product in Deligne cohomology implies that $c_1(f) \cup c_1(g)$ is represented by the element of

$$H_{\mathcal{D}}^2(\eta_X, \mathbb{Z}(2)) \approx H^1(\eta_X, \mathbb{C}/\mathbb{Z}(2))$$

defined by

$$\gamma \mapsto I(f, g, \gamma). \quad \square$$

We now globalize this construction. For each $x \in X$, there is map

$$\delta_x : H^1(\eta_X, \mathbb{C}^*) \rightarrow \mathbb{C}^*.$$

To define $\delta_x(l)$, represent l by a flat line bundle $L \rightarrow U$ over a Zariski open subset U of X . Chose a small closed disk $\overline{\Delta}$ in X , centered at x , such that $\overline{\Delta} - \{x\}$ is contained in U . Define $\delta(l)$ to be the monodromy of L about the boundary of $\overline{\Delta}$.

Suppose that $\nu : F^\times \rightarrow \mathbb{Z}$ is a valuation on a field F . Let \mathcal{O} be the associated valuation ring (i.e., 0 and those elements of F^\times with valuation ≥ 0). Let \mathfrak{P} be the maximal ideal of \mathcal{O} . The tame symbol of $f, g \in F^\times$ is defined by

$$(f, g)_\nu = (-1)^{\nu(f)\nu(g)} \frac{f^{\nu(g)}}{g^{\nu(f)}} \bmod \mathfrak{P}$$

(See, e.g., [30].) To each $x \in X$ associate the valuation which takes a function f to its order $\nu_x(f)$ at x . In this way we associate a tame symbol $(\ , \)_x$ to each $x \in X$.

Proposition 6.6 *Suppose that $x \in X$. If $f, g \in \mathbb{C}(X)$, then*

$$\delta_x \langle f, g \rangle = (f, g)_x.$$

Proof. Since both sides of the expression in the statement of the proposition are skew symmetric and bilinear, we can reduce, with the help of (6.2), to the following 2 cases. First, if z is a local holomorphic parameter about x , then we have to show that $\delta_x \langle z, z \rangle = -1$. Second, if f is a unit in a neighbourhood of x , then

$$\delta_x \langle f, g \rangle = f^{\nu_x(g)}(x)$$

From (6.3) and (5.1)(ii) it follows that

$$I(z, z, p) = \int_{\partial \Delta} \frac{dz}{z} \frac{dz}{z} = \frac{1}{2} \left(\int_{\partial \Delta} \frac{dz}{z} \right)^2 = \frac{(2\pi i)^2}{2} \in \mathbb{C}/\mathbb{Z}(2),$$

where Δ is a sufficiently small imbedded disk in X centered at x . Under the standard isomorphism $\mathbb{C}/\mathbb{Z}(2) \approx \mathbb{C}^*$, this corresponds to $e^{i\pi} = -1$. This proves the first assertion.

To prove the second, write $f = e^\phi$, where ϕ is holomorphic in a neighbourhood of x . By (6.3), we have

$$I(f, g, p) = \int_{\partial\Delta} (\phi(z) - \phi(p)) \frac{dg}{g} + \phi(p) \int_{\partial\Delta} \frac{dg}{g} = \int_{\partial\Delta} \phi(z) \frac{dg}{g}.$$

By the Residue Theorem this equals $2\pi i \nu_x(g) \phi(x)$. So

$$\delta_x \langle f, g \rangle = \exp(\nu_x(g) \phi(x)) = f(x)^{\nu_x(g)}.$$

This proves the second assertion. \square

The following version of the Gysin sequence is easily verified.

Proposition 6.7 *The sequence*

$$0 \rightarrow H^1(X, \mathbb{C}^*) \rightarrow H^1(\eta_X, \mathbb{C}^*) \xrightarrow{\oplus \delta_x} \bigoplus_{x \in X} \mathbb{C}^*$$

is exact. \square

The analogue of the Gysin sequence in algebraic K -theory is the localization sequence (reference !). In our case, it asserts that the sequence

$$K_2(X) \rightarrow K_2(\eta_X) \xrightarrow{\oplus (\cdot, \cdot)_x} \bigoplus_{x \in X} \mathbb{C}^*$$

is exact. By (6.6), the diagram

$$\begin{array}{ccccccc} K_2(X) & \rightarrow & K_2(\eta_X) & \xrightarrow{\oplus (\cdot, \cdot)_x} & \bigoplus_{x \in X} \mathbb{C}^* \\ & & \downarrow c_2 & & \parallel \\ 0 \rightarrow H^1(X, \mathbb{C}^*) & \rightarrow & H^1(\eta_X, \mathbb{C}^*) & \xrightarrow{\oplus \delta_x} & \bigoplus_{x \in X} \mathbb{C}^* \end{array}$$

commutes. Since the rows are exact, the Chern class c_2 induces a map $K_2(X) \rightarrow H^1(X, \mathbb{C}^*)$ which must be the Chern class by naturality.

This construction extends easily to give a description of the regulator

$$c_2 : K_2(X) \rightarrow H_{\mathcal{D}}^2(X, \mathbb{Z}(2))$$

where X is a smooth variety over \mathbb{C} . The construction proceeds in the same way, except that the line bundles are no longer flat. For a mixed Hodge structure H denote $\text{Hom}_{\text{Hodge}}(\mathbb{Z}, H)$, the set of ‘‘Hodge classes’’ in H of type $(0, 0)$, by ΓH .

Proposition 6.8 *If X is smooth over \mathbb{C} , then there is a natural isomorphism between $H_D^2(X, \mathbb{Z}(2))$ and the group which consists of the pairs (L, ∇) , where L is a holomorphic line bundles over X and ∇ is a holomorphic connection whose curvature times $2\pi i$ lies in $\Gamma H^2(X, \mathbb{Z}(2))$. \square*

7 The polylogarithm variation of mixed Hodge structure

Let X be a smooth complex algebraic curve and \overline{X} a smooth compactification of it. Let $D = \overline{X} - X$. Recall from [36] that a *variation of mixed Hodge structure* over X consists of

1. a \mathbb{Q} local system $\mathbb{V} \rightarrow X$ which has a filtration by local systems

$$\cdots \subseteq \mathbb{W}_{l-1} \subseteq \mathbb{W}_l \subseteq \mathbb{W}_{l+1} \subseteq \cdots$$

which exhausts \mathbb{V} and whose intersection is 0. We shall denote the fiber of \mathbb{V} over $x \in X$ by V_x and the fiber of \mathbb{W}_l by $W_l V_x$. We will also assume that each local monodromy operator $T_P : V_P \rightarrow V_P$, about each $P \in D$, is unipotent.

2. a Hodge filtration

$$\cdots \supseteq \mathcal{F}^{p-1} \supseteq \mathcal{F}^p \supseteq \mathcal{F}^{p+1} \supseteq \cdots$$

of the corresponding holomorphic vector bundle $\mathcal{V} := \mathbb{V} \otimes_{\mathbb{Q}} \mathcal{O}_X$ by holomorphic sub-bundles. These are required to satisfy Griffiths' transversality: If

$$\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes_{\mathcal{O}_X} \Omega_X^1$$

is the natural flat connection, then

$$\nabla(\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes_{\mathcal{O}_X} \Omega_X^1.$$

Denote the fiber of \mathcal{F}^p over $x \in X$ by $F^p V_x$.

3. For each $x \in X$, the filtrations $W_{\bullet} V_x$ and $F^{\bullet} V_x$ define a mixed Hodge structure on V_x .

4. Denote Deligne's canonical extension of \mathcal{V} to \overline{X} by $\overline{\mathcal{V}} \rightarrow \overline{X}$ [10]. The Hodge bundles \mathcal{F}^p are required to extend to holomorphic sub-bundles $\overline{\mathcal{F}}^p$ of $\overline{\mathcal{V}}$. (Note that the weight bundles $\mathcal{W}_l := \mathbb{W}_l \otimes_{\mathbb{Q}} \mathcal{O}_X$ automatically extend to sub-bundles $\overline{\mathcal{W}}$ of $\overline{\mathcal{V}}$ as they are flat.)
5. about each point $P \in D$, there is a relative weight filtration [36]. This is an important condition which is rather technical in general. However, in the case where the global monodromy representation

$$\rho_x : \pi_1(X, x) \rightarrow GL(V_x)$$

is unipotent, the condition reduces to the much simpler condition

$$N_P(W_l V_x) \subseteq W_{l-2} V_x$$

for each $P \in D$, where N_P is the local monodromy logarithm

$$N_P = \frac{1}{2\pi i} \log T_P.$$

(See [24].)

Theorem 7.1 *The n th polylogarithm local system underlies a good variation of mixed Hodge structure whose weight graded quotients are canonically isomorphic to $\mathbb{Q}, \mathbb{Q}(1), \dots, \mathbb{Q}(n)$.*

Proof. Let $\mathbb{V} \rightarrow \mathbb{C} - \{0, 1\}$ be the n th polylogarithm local system, and \mathcal{V} the corresponding holomorphic vector bundle. By (2.5), the canonical extension of this to \mathbb{P}^1 is the trivial bundle

$$\mathbb{P}^1 \times \mathbb{C}^{n+1} \rightarrow \mathbb{P}^1.$$

Denote the standard basis of \mathbb{C}^{n+1} by e_0, e_1, \dots, e_n . The fiber V_x is the \mathbb{Q} linear span of $\lambda_0(x), \lambda_1(x), \dots, \lambda_n(x)$, the rows of $\Lambda(x)$. Define

$$W_{-2l+1} \mathbb{C}^{n+1} = W_{-2l} \mathbb{C}^{n+1} = \text{span} \{e_l, \dots, e_n\} \quad (11)$$

and

$$F^{-p} \mathbb{C}^{n+1} = \text{span} \{e_0, \dots, e_p\}. \quad (12)$$

Define

$$\overline{\mathcal{F}}^p = \mathbb{P}^1 \times F^p \mathbb{C}^{n+1} \subseteq \overline{\mathcal{V}} \text{ and } \overline{\mathcal{W}}_l = \mathbb{P}^1 \times W_l \mathbb{C}^{n+1} \subseteq \overline{\mathcal{V}}.$$

Observe that the weight filtration comes from a filtration defined on \mathbb{V} :

$$W_{-2l+1}V_x = W_{-2l}V_x = \text{span} \{ \lambda_l(x), \dots, \lambda_n(x) \} \quad \square$$

Suppose that $\mathbb{V} \rightarrow X$ is a good variation of mixed Hodge structure with unipotent monodromy about each point of $D = \overline{X} - X$. Let $P \in D$. For each non-zero tangent vector \vec{v} of X at p , there is a canonical mixed Hodge structure on $V_{\mathbb{C}}$, the fiber of $\overline{\mathcal{V}}$ over P . This is called the *limit mixed Hodge structure associated to \vec{v}* . The Hodge and weight filtrations on $V_{\mathbb{C}}$ are defined by letting $F^p V_{\mathbb{C}}$ and $W_l V_{\mathbb{C}}$ be the fibers of $\overline{\mathcal{F}}^p$ and $\overline{\mathcal{W}}_l$ over P , respectively. To construct the limit mixed Hodge structure, we have to construct a rational form $V_{\mathbb{Q}}$ of $V_{\mathbb{C}}$ and show that the weight filtration defined above is the complexification of a filtration of $V_{\mathbb{Q}}$.

To construct $V_{\mathbb{Q}}$, choose an imbedded closed disk $\overline{\Delta}$ in \overline{X} centered at P . Let t be a holomorphic parameter in $\overline{\Delta}$ such that $t(P) = 0$ and $|t| = 1$ is $\partial\overline{\Delta}$. By choosing the disk to be small enough, we may suppose that $\overline{\Delta} - \{0\} \subseteq X$.

We first consider the case where $\vec{v} = \partial/\partial t$. Let $x \in \overline{\Delta}$ be the point corresponding to $t = 1$. Choose a \mathbb{Q} basis v_1, \dots, v_m of V_x , the fiber of \mathbb{V} over x . Let $v_1(t), \dots, v_m(t)$ be flat (possibly multivalued) sections of \mathbb{V} over $\overline{\Delta}^*$ which satisfy $v_j(1) = v_j$ for each j . Let $T : V_x \rightarrow V_x$ be the local monodromy operator, and $N = \log T / 2\pi i$ be the local monodromy logarithm. For each j , define

$$s_j(t) = t^{-N} v_j(t).$$

Then each $s_j(t)$ is a single valued section of $\overline{\mathcal{V}}$ over $\overline{\Delta}^*$. In fact, by the construction of the canonical extension $\overline{\mathcal{V}} \rightarrow \overline{X}$, the s_j comprise a local framing of $\overline{\mathcal{V}}$ over $\overline{\Delta}$. In particular, $s_1(0), \dots, s_m(0)$ is a \mathbb{C} basis of $V_{\mathbb{C}}$. Define the rational form $V_{\mathbb{Q}}$ of $V_{\mathbb{C}}$ which corresponds to $\partial/\partial t$ to be the \mathbb{Q} -linear span of $s_1(0), \dots, s_m(0)$. By choosing the basis v_1, \dots, v_m of V_x to be adapted to its weight filtration, one can easily show that the weight filtration of $V_{\mathbb{C}}$ is the complexification of a filtration of $V_{\mathbb{Q}}$. The \mathbb{Q} structure on $V_{\mathbb{C}}$ which corresponds to the tangent vector $\vec{v} = \lambda \partial/\partial t$ is defined to be

$$V_{\mathbb{Q}}(\vec{v}) = \lambda^N V_{\mathbb{Q}}.$$

It is not difficult to show that this rational structure depends only on the tangent vector \vec{v} , and not on the choice of the parameter t .

If the weight graded quotients of $\mathbb{V} \rightarrow X$ are constant as variations of Hodge structure (e.g. the polylogarithm variations), it is not difficult to show that $((V_{\mathbb{Q}}(\vec{v}), W_{\bullet}), (V_{\mathbb{C}}, F^{\bullet}))$ is a mixed Hodge structure, and that $N : V \rightarrow V(-1)$ is a morphism of mixed Hodge structures.

The following result is due to Deligne [13] in the case $n = 2$. It is a straight forward computation using the monodromy computation (2.2) and the procedure described above.

Theorem 7.2 *Let z be the natural coordinate function on $\mathbb{C} - \{0, 1\}$. The limit mixed Hodge structure on the n th polylogarithm variation at the tangent vector $\partial/\partial z$ at 0 has rational structure spanned by the vectors*

$$\begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{pmatrix} = \left(\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 2\pi i & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & (2\pi i)^n \end{array} \right) \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{pmatrix}$$

The limit mixed Hodge structure associated with the tangent vector $-\partial/\partial z$ at 1 has rational structure spanned by the vectors

$$\begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{pmatrix} = \left(\begin{array}{c|cccc} 1 & 0 & \zeta(2) & \cdots & \zeta(n) \\ \hline 0 & 2\pi i & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & (2\pi i)^n \end{array} \right) \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_n \end{pmatrix}$$

where $\zeta(s)$ denotes the Riemann zeta function. In both cases, the Hodge and weight filtrations are defined as in (11) and (12).

8 Mixed Hodge structure on π_1

In this section we give the construction of the mixed Hodge structure on the fundamental group $\pi_1(X, x)$ in the special case where X is a smooth variety over \mathbb{C} whose $H^1(X)$ is pure of weight 2. This purity condition may be

restated in several equivalent ways. For example, $H^1(X)$ is pure of weight 2 if and only if one and (and hence all) smooth completions \overline{X} of X have first Betti number 0. In particular, all Zariski open subsets of a Grassmannians have this property.

To put a mixed Hodge structure on $\pi_1(X, x)$, it is necessary to linearize it. The first step is to replace the fundamental group by its group algebra $\mathbb{Q}\pi_1(X, x)$. This object is not well enough behaved, and we need to linearize it further, which we do by completion. Let J be the augmentation ideal. That is, J is the kernel of the augmentation

$$\epsilon : \mathbb{Q}\pi_1(X, x) \rightarrow \mathbb{Q}$$

which takes each $g \in \pi_1(X, x)$ to 1. The powers of J define a topology on $\mathbb{Q}\pi_1(X, x)$, and we consider

$$\mathbb{Q}\pi_1(X, x)^\wedge = \varinjlim \mathbb{Q}\pi_1(X, x)/J^m,$$

the J -adic completion.

The completed group ring is a complete Hopf algebra. That is, it is a topological algebra and has a continuous algebra homomorphism

$$\Delta : \mathbb{Q}\pi_1(X, x)^\wedge \rightarrow \mathbb{Q}\pi_1(X, x)^\wedge \hat{\otimes} \mathbb{Q}\pi_1(X, x)^\wedge$$

Here $\hat{\otimes}$ denotes completed tensor product. This algebra homomorphism is induced by the usual diagonal

$$\mathbb{Q}\pi_1(X, x) \rightarrow \mathbb{Q}\pi_1(X, x) \otimes \mathbb{Q}\pi_1(X, x)$$

which takes each $g \in \mathbb{Q}\pi_1(X, x)$ to $g \otimes g$.

The Malcev Lie algebra $\mathfrak{g}(X, x)$ associated to $\pi_1(X, x)$ is, by definition, the set of primitive elements

$$\{X \in \mathbb{Q}\pi_1(X, x)^\wedge : \Delta X = 1 \otimes X + X \otimes 1\}$$

of $\mathbb{Q}\pi_1(X, x)^\wedge$. This is a complete topological Lie algebra. The bracket of the elements A, B of \mathfrak{g} is their commutator $AB - BA$, which is again primitive. The topology is induced from that of $\mathbb{Q}\pi_1(X, x)^\wedge$. The completed group ring may be recovered from its primitive elements as its completed enveloping algebra $U^\wedge \mathfrak{g}(X, x)$ (see [31, Appendix A]).

A mixed Hodge structure on $\mathbb{Q}\pi_1(X, x)^\wedge$ is, by definition, a compatible sequence of mixed Hodge structures

$$\cdots \rightarrow \mathbb{Q}\pi_1(X, x)/J^3 \rightarrow \mathbb{Q}\pi_1(X, x)/J^2 \rightarrow \mathbb{Q}\pi_1(X, x)/J \rightarrow \mathbb{Q} \rightarrow 0$$

on the truncations of the group ring. Note that each of these is a finite dimensional vector space.

If the product and diagonal of $\mathbb{Q}\pi_1(X, x)^\wedge$ are morphisms of mixed Hodge structure, then, as $\mathfrak{g}(X, x)$ is the kernel of the reduced diagonal

$$\overline{\Delta} : \mathbb{Q}\pi_1(X, x)^\wedge \rightarrow [\mathbb{Q}\pi_1(X, x)^\wedge/\mathbb{Q}] \hat{\otimes} [\mathbb{Q}\pi_1(X, x)^\wedge/\mathbb{Q}] \approx J \hat{\otimes} J,$$

the Malcev Lie algebra inherits a mixed Hodge structure compatible with its Lie algebra structure. This mixed Hodge structure determines the one on $\mathbb{Q}\pi_1(X, x)^\wedge$ as $\mathfrak{g}(X, x)$ generates $\mathbb{Q}\pi_1(X, x)^\wedge$ topologically.

Theorem 8.1 [28],[21] *If (X, x) is a complex algebraic variety, then $\mathbb{Q}\pi_1(X, x)^\wedge$ and $\mathfrak{g}(X, x)$ have canonical mixed Hodge structures which are compatible with their algebraic structures.*

It is important to note that if $H^1(X, \mathbb{Q})$ is non-trivial, then this mixed Hodge structure depends non-trivially on the basepoint $x \in X$ (cf. [20, §§6,7]).

We will sketch the construction of this mixed Hodge structure in the case when $H^1(X)$ is pure of weight 2. Write $X = \overline{X} - D$, where \overline{X} is smooth and complete, and D is a divisor with normal crossings in \overline{X} . For convenience, we set

$$\Omega^\bullet(X) = H^0(\overline{X}, \Omega_{\overline{X}}^\bullet(\log D)).$$

By results of Deligne [11, (3.2.14)], every element of $\Omega^\bullet(X)$ is closed and the obvious map

$$\Omega^\bullet(X) \rightarrow H^\bullet(X, \mathbb{C})$$

is injective. The tensor algebra

$$T := \bigoplus_{n \in \mathbb{N}} \Omega^1(X)^{\otimes (-n)}$$

on the dual of $\Omega^1(X)$ is a Hopf algebra; the diagonal is defined as the algebra homomorphism which takes each $X \in \Omega^1(X)^*$ to $1 \otimes X + X \otimes 1$. The ideal $(\text{im}\delta)$ in T generated by the image of the dual of the cup product

$$\delta : \Omega^2(X)^* \rightarrow \Lambda^2 \Omega^1(X)^* \subseteq \Omega^1(X)^{\otimes(-2)}$$

is a Hopf ideal. It follows that

$$A := T/(\text{im}\delta)$$

is a Hopf algebra whose diagonal is induced from that of the tensor algebra. The set of primitive elements of A is

$$PA = \mathbb{L}(\Omega^1(X)^*)/(\text{im}\delta),$$

where $\mathbb{L}(V)$ denotes the free Lie algebra generated by the vector space V .

Denote the ideal generated by $\Omega^1(X)^*$ by I . The powers of I define a topology on A . Denote the I -adic completion of A by A^\wedge . The set of primitive elements of A^\wedge is the I -adic completion of PA .

The following result is a special case of a theorem of K.-T. Chen [9, (3.5)].

Proposition 8.2 *For each $x \in X$ there is a canonical isomorphism*

$$\Theta_x : \mathbb{C}\pi_1(X, x)^\wedge \rightarrow A^\wedge$$

of complete Hopf algebras.

Proof. Let $\omega \in \Omega^1(X) \otimes \Omega^1(X)^*$ be the element which corresponds to the identity $\Omega^1(X) \rightarrow \Omega^1(X)$. This can be viewed as an element of $\Omega^1(X) \otimes PA^\wedge$. This form is integrable. That is,

$$d\omega + \omega \wedge \omega = 0.$$

It follows that the value of the A -valued iterated integral

$$1 + \int \omega + \int \omega\omega + \int \omega\omega\omega + \cdots$$

on each path in X depends only on its homotopy class relative to its end points (cf. [20, §3], for example). It follows from this and (5.1)(ii) that this

map induces a well defined homomorphism from $\pi_1(X, x)$ into the group of units of A^\wedge . This extends to an algebra homomorphism

$$\mathbb{C}\pi_1(X, x) \rightarrow A^\wedge.$$

Since the augmentation ideal of $\mathbb{C}\pi_1(X, x)$ is mapped into the ideal I of A^\wedge , it follows that this homomorphism extends to a continuous algebra homomorphism

$$\Theta_x : \mathbb{C}\pi_1(X, x)^\wedge \rightarrow A^\wedge.$$

The property (5.1)(iv) implies that Θ_x commutes with the diagonals; that is, Θ_x is a Hopf algebra homomorphism.

The graded module associated to the filtration of $\mathbb{C}\pi_1(X, x)^\wedge$ by powers of J is generated by J/J^2 , and this is isomorphic to $H_1(X)$. Similarly, the graded module associated to the filtration of A^\wedge by the powers of I is generated by I/I^2 , which is also isomorphic to $H_1(X)$. The map Θ_x induces an isomorphism $J/J^2 \approx I/I^2$. Since both algebras are complete, this implies that Θ_x is surjective. One can show, without too much difficulty, that the map $J^2/J^3 \rightarrow I^2/I^3$ induced by Θ_x is also an isomorphism (cf. [20, (6.1)]). It is then relatively straightforward to show that Θ_x must be injective. The idea is that A^\wedge contains no other relations other than those that are consequences of the quadratic ones, while $\mathbb{C}\pi_1(X, x)^\wedge$ has at least these relations. Since Θ_x is well defined, it must be an isomorphism. \square

The next step in constructing the mixed Hodge structure on $\mathbb{Q}\pi_1(X, x)^\wedge$ is to define the Hodge and weight filtrations. We do this by defining them on A^\wedge and transferring them to $\mathbb{Q}\pi_1(X, x)^\wedge$ via the isomorphism Θ_x .

The ring A is graded as the ideal $(\text{im}\delta)$ is graded. Write

$$A = \bigoplus_{n \in \mathbb{N}} A_n$$

where A_n is the image of $\Omega^1(X)^{\otimes(-n)}$ in A . The assumption that $H^1(X)$ be pure of weight 2 implies that it is has Hodge type $(1, 1)$. Consequently, $\Omega^1(X)^* \approx H_1(X)$ has Hodge type $(-1, -1)$. It is therefore natural to define Hodge and weight filtrations on A by

$$F^{-p}A = \bigoplus_{n \leq p} A_n$$

and

$$W_{-m}A = \bigoplus_{n \geq m/2} A_n$$

Now define Hodge and weight filtrations on $\mathbb{Q}\pi_1(X, x)/J^l$ by transferring the Hodge and weight filtrations from A/I^l via the isomorphism

$$\Theta_x : \mathbb{Q}\pi_1(X, x)/J^l \rightarrow A/I^l.$$

This data defines a mixed Hodge structure on $\mathbb{Q}\pi_1(X, x)/J^l$. To see this, first observe that

$$Gr_m^W \mathbb{Q}\pi_1(X, x)/J^l = \begin{cases} J^r/J^{r+1} & \text{when } m = -2r \text{ and } 0 \leq r < l; \\ 0 & \text{otherwise.} \end{cases}$$

The Hodge filtration induced on Gr_{-2p}^W satisfies

$$F^{-p} [J^p/J^{p+1}] = J^p/J^{p+1}, \quad F^{-p+1} [J^p/J^{p+1}] = 0$$

when $0 \leq p < l$. Because the weight filtration is defined over \mathbb{Q} , it follows that these filtrations define a mixed Hodge structure on $\mathbb{Q}\pi_1(X, x)/J^l$ whose weight graded quotients are all of even weight, and where the $2p$ th graded quotient is of type (p, p) . Since the multiplication and comultiplication of A preserve the filtrations and are defined over \mathbb{Q} , they are morphisms of mixed Hodge structure. It follows that $\mathfrak{g}(X, x)$, endowed with the induced filtrations, is a mixed Hodge structure.

We conclude this section by relating this mixed Hodge structure to unipotent variations of mixed Hodge structure. A variation of mixed Hodge structure over a smooth variety X is good if its restriction to every curve satisfies the conditions in §7. A good variation of mixed Hodge structure $\mathbb{V} \rightarrow X$ over a smooth variety X is *unipotent* if one (and hence all) monodromy representations

$$\rho_x : \pi_1(X, x) \rightarrow \text{Aut } V_x \tag{13}$$

are unipotent. This condition is equivalent to the condition that each of the variations of Hodge structure $Gr_m^W \mathbb{V}$ be constant.

The monodromy representation (13) induces a map

$$\theta_x : \mathbb{Q}\pi_1(X, x) \rightarrow \text{End } V_x.$$

Since the representation is unipotent, there exists l such that J^l is contained in $\ker \theta_x$. It follows that there is an algebra homomorphism

$$\theta_x : \mathbb{Q}\pi_1(X, x)/J^l \rightarrow \text{End} V_x. \quad (14)$$

Both sides of this last equation have natural mixed Hodge structures.

Theorem 8.3 [24] *For each $x \in X$, the representation (14) is a morphism of mixed Hodge structures.*

Define the category of Hodge theoretic representations of $\pi_1(X, x)$ to be the set of pairs (V, ρ) , where V is a mixed Hodge structure and ρ is a unipotent representation $\pi_1(X, x) \rightarrow \text{Aut} V$ which induces a morphism of mixed Hodge structure

$$\theta_x : \mathbb{Q}\pi_1(X, x)/J^l \rightarrow \text{End} V$$

for l sufficiently large. Theorem 8.3 implies that taking the fiber at x defines a functor from the category of unipotent variations of mixed Hodge structure over X to the category of Hodge theoretic representations of $\pi_1(X, x)$.

Theorem 8.4 [24] *This functor is an equivalence of categories.*

The proofs of Theorems 8.3 and 8.4 in the case when $H^1(X)$ is pure of weight 2 are considerably simpler than in general case. (See [25] for a proof in this case.)

A good variation of mixed Hodge structure \mathbb{V} over a smooth variety X is called a *Tate variation of mixed Hodge structure* if all of its weight graded quotients are constant and of even weight, and if each of the variations $Gr_{2p}^W \mathbb{V}$ is of type (p, p) . The polylogarithm variations are examples of Tate variations of mixed Hodge structure. Now suppose that \overline{X} is any smooth compactification of X where $D = \overline{X} - X$ is a divisor with normal crossings in \overline{X} . Let $\overline{\mathbb{V}} \rightarrow \overline{X}$ be the canonical extension of \mathbb{V} to \overline{X} . The following result is a simple consequence of Theorem 8.3 and [19, (6.4)].

Theorem 8.5 *If $\mathbb{V} \rightarrow X$ is a Tate variation of mixed Hodge structure, then its canonical extension $\overline{\mathbb{V}} \rightarrow \overline{X}$ is trivial as a holomorphic vector bundle, so*

that there is a complex vector space V and a bundle isomorphism

$$\begin{array}{ccc} \overline{\mathcal{V}} & \rightarrow & V \times \overline{X} \\ \downarrow & & \downarrow \\ \overline{X} & = & \overline{X} \end{array}$$

Moreover, there are filtrations F^\bullet and W_\bullet of V such that the extended Hodge and weight bundles $\overline{\mathcal{F}}^p$ and $\overline{\mathcal{W}}_l$ correspond to $F^p \times \overline{X}$ and $W_l \times \overline{X}$, respectively, under the bundle isomorphism. \square

One important example of a unipotent variation of mixed Hodge structure over a smooth variety X is the one whose fiber over $x \in X$ is the truncated group ring $\mathbb{Q}\pi_1(X, x)/J^l$. This is a good variation because the monodromy representation

$$\mathbb{Q}\pi_1(X, x)/J^l \rightarrow \text{Aut} \mathbb{Q}\pi_1(X, x)/J^l$$

can be written in terms of left and right multiplication, and is thus a morphism of mixed Hodge structure. Such variations form an inverse system of variations, and we call the inverse limit the *tautological variation* over X . In case when $H^1(X)$ has weight 2, this variation is a Tate variation of mixed Hodge structure, and can be described explicitly. View A^\wedge as a subalgebra of $\text{End } A^\wedge$ via the right regular representation.

Proposition 8.6 *If $H^1(X)$ is of weight 2, then the tautological variation over X has canonical extension $A^\wedge \times \overline{X} \rightarrow \overline{X}$. The connection form of the canonical flat connection on this bundle is given by the PA^\wedge valued 1-form*

$$\omega \in \Omega^1(X) \otimes H_1(X) \subseteq \Omega^1(X) \otimes PA^\wedge \subseteq \Omega^1(X) \otimes \text{End } A^\wedge$$

which corresponds to the canonical isomorphism $\Omega^1(X) \approx H^1(X)$. The extended Hodge and weight bundles are $F^p A^\wedge \times \overline{X}$ and $W_l A^\wedge \times \overline{X}$. \square

9 Hodge theoretic interpretation of regulators

In this section we give a Hodge theoretic interpretation of the regulators constructed in Sections 4 and 6. These interpretations are due to Deligne [13].

Throughout this section, Λ will denote \mathbb{Z}, \mathbb{Q} or \mathbb{R} . Suppose that $V = (V_\Lambda, (V_{\Lambda \otimes \mathbb{Q}}, W_\bullet), (V_{\mathbb{C}}, F^\bullet))$ is a mixed Hodge structure where the underlying lattice V_Λ is torsion free. The ring of endomorphisms $\text{End} V$ has a mixed Hodge structure whose Hodge and weight filtrations are defined by

$$F^p \text{End}_{\mathbb{C}} V = \{ \phi \in \text{End}_{\mathbb{C}} V : \phi(F^q V) \subseteq F^{p+q} V \}$$

and

$$W_l \text{End}_{\mathbb{Q}} V = \{ \phi \in \text{End}_{\Lambda \otimes \mathbb{Q}} V : \phi(W_m V) \subseteq W_{m+l} V \}.$$

Set $\mathfrak{g} = W_{-1} \text{End} V$. This is a nilpotent Lie algebra with a mixed Hodge structure—the bracket being the commutator $[\phi, \psi] = \phi\psi - \psi\phi$. The subspace $F^0 \mathfrak{g}$ is a Lie sub-algebra. Denote the simply connected Lie groups which correspond to $\mathfrak{g}_{\mathbb{C}}$ and $F^0 \mathfrak{g}$ by G and $F^0 G$, respectively. These are unipotent subgroups of $\text{Aut}_{\mathbb{C}} V$. Set $G_\Lambda = G \cap \text{End}_\Lambda V$. We view G_Λ as acting on the right of V_Λ .

For each $g \in G$, the triple $(V_\Lambda g, (V_{\Lambda \otimes \mathbb{Q}} g, W_\bullet g), (V_{\mathbb{C}}, F^\bullet))$ is a mixed Hodge structure whose weight graded quotients are canonically isomorphic to those of V . It is not difficult to show that every mixed Hodge structure with torsion free lattice and weight graded quotients canonically isomorphic to those of V can be constructed this way. More generally, we have the following result which is easily proved (cf. [7]).

Proposition 9.1 *The set of Λ -mixed Hodge structures whose weight graded quotients are canonically isomorphic those of V is naturally isomorphic to*

$$G_\Lambda \backslash G / F^0 G.$$

The double coset of $g \in G$ corresponds to the mixed Hodge structure

$$V = (V_\Lambda g, (V_{\Lambda \otimes \mathbb{Q}} g, W_\bullet g), (V_{\mathbb{C}}, F^\bullet)). \quad \square$$

This identification can be used to compute extension groups of mixed Hodge structures. Suppose that A and B are Λ -Hodge structures whose underlying Λ module is torsion free. Suppose that the weight of A is greater than that of B . If we take $V = A \oplus B$ then, for $R = \Lambda, \mathbb{C}$,

$$G_R = \text{Hom}_R(A, B)$$

In this case the moduli space of mixed Hodge structures with weight graded quotients canonically isomorphic to A and B is the group $\text{Ext}_{\mathcal{H}}^1(A, B)$ of extensions of A by B in the category \mathcal{H} of mixed Hodge structures. Applying Proposition (9.1) to the split mixed Hodge structure $A \oplus B$, we obtain the well known formula for $\text{Ext}_{\mathcal{H}}^1$ (cf. [7]).

Proposition 9.2 *With A and B as above, there is a canonical isomorphism*

$$\text{Ext}_{\mathcal{H}}^1(A, B) \approx \frac{\text{Hom}_{\mathbb{C}}(A, B)}{\text{Hom}_{\Lambda}(A, B) + F^0 \text{Hom}_{\mathbb{C}}(A, B)}.$$

An important special case is where $A = \mathbb{Z}$, $B = \mathbb{Z}(n)$ and $n \geq 1$. (Recall that $\mathbb{Z}(n)$ is the Hodge structure of type $(-n, -n)$ whose lattice is the subgroup $(2\pi i)^n \mathbb{Z}$ of \mathbb{C} .) In this case we have

$$\text{Ext}_{\mathcal{H}}^1(\mathbb{Z}, \mathbb{Z}(n)) \approx \mathbb{C}/\mathbb{Z}(n).$$

Following through the construction, we see that the mixed Hodge structure which corresponds to $\lambda \in \mathbb{C}/\mathbb{Z}(n)$ can be described as follows. Denote the standard basis of \mathbb{C}^2 by e_0, e_n . These have type $(0, 0), (-n, -n)$, respectively. The Hodge and weight filtrations on \mathbb{C}^2 are defined by

$$W_l \mathbb{C}^2 = \text{span}\{e_j : -j \leq l\}$$

and

$$F^p \mathbb{C}^2 = \text{span}\{e_j : -j \geq p\}.$$

The mixed Hodge structure which corresponds to λ has integral basis the two vectors

$$\begin{pmatrix} 1 & \lambda \\ 0 & (2\pi i)^n \end{pmatrix} \begin{pmatrix} e_0 \\ e_n \end{pmatrix}.$$

In particular, the extension of \mathbb{Z} by $\mathbb{Z}(1)$ which corresponds to $x \in \mathbb{C}^* \approx \mathbb{C}/\mathbb{Z}(1)$ has integral basis spanned by the vectors

$$\begin{pmatrix} 1 & \log x \\ 0 & 2\pi i \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \end{pmatrix}.$$

A unipotent variation of mixed Hodge structure over a smooth variety X whose weight graded quotients are canonically isomorphic to \mathbb{Z} and $\mathbb{Z}(m)$, ($m \geq 1$) will determine a *classifying map* $X \rightarrow \text{Ext}_{\mathcal{H}}^1(\mathbb{Z}, \mathbb{Z}(m))$.

Proposition 9.3 *When $m > 1$ the classifying map is constant. When $m = 1$, a map*

$$X \rightarrow \mathrm{Ext}_{\mathcal{H}}^1(\mathbb{Z}, \mathbb{Z}(1)) \approx \mathbb{C}^*$$

is the classifying map of a good variation of mixed Hodge structure if and only if it is an algebraic function on X .

Proof. In both cases, the canonical extension of the variation to a good compactification \overline{X} of X is a trivial bundle (8.5), as are the extended Hodge and weight bundles. In the first case, Griffiths' transversality forces the integral lattice to be constant. In the second, the regularity of the connection of the canonical extension corresponds to the classifying map $X \rightarrow \mathbb{C}^*$ of the variation having poles at infinity. \square

Since there is a canonical isomorphism

$$\mathrm{Ext}_{\mathcal{H}}^1(\Lambda, \Lambda(m)) \approx \mathbb{C}/\Lambda(m),$$

the regulator $K_m(\mathbb{C}) \rightarrow \mathbb{C}/\Lambda(m)$ can then be interpreted as a map

$$K_m(\mathbb{C}) \rightarrow \mathrm{Ext}_{\mathcal{H}}^1(\Lambda, \Lambda(m)).$$

A motivic description of this regulator in the case when $m = 3$ is given in [3].

The regulator

$$c_2 : K_2(X) \rightarrow H_{\mathcal{D}}^2(X, \mathbb{Z}(2))$$

also admits a Hodge theoretic interpretation. This time we take our reference Hodge structure V to be the direct sum of $\mathbb{Z}(0)$, $\mathbb{Z}(1)$ and $\mathbb{Z}(2)$. The moduli space of mixed Hodge structures whose weight graded quotients are canonically isomorphic to these Hodge structures is

$$H_{\mathbb{Z}} \backslash H_{\mathbb{C}}.$$

where H denotes the Heisenberg group defined in §6. The bundle projection

$$H_{\mathbb{Z}} \backslash H_{\mathbb{C}} \rightarrow \mathbb{C}^* \times \mathbb{C}^*$$

may be interpreted as the map which takes a mixed Hodge structure $V \in H_{\mathbb{Z}} \backslash H_{\mathbb{C}}$ to

$$(V/\mathbb{Z}(2), W_2 V) \in \mathrm{Ext}_{\mathcal{H}}^1(\mathbb{Z}, \mathbb{Z}(1)) \times \mathrm{Ext}_{\mathcal{H}}^1(\mathbb{Z}(1), \mathbb{Z}(2)) \approx \mathbb{C}^* \times \mathbb{C}^*.$$

We next consider the problem of determining which maps $X \rightarrow H_{\mathbb{Z}} \backslash H_{\mathbb{C}}$ classify variations of mixed Hodge structure.

Proposition 9.4 *A function $f : X \rightarrow H_{\mathbb{Z}} \backslash H_{\mathbb{C}}$ is the classifying map of a variation of mixed Hodge structure over X with weight graded quotients canonically isomorphic to $\mathbb{Z}, \mathbb{Z}(1)$ and $\mathbb{Z}(2)$ if and only if*

1. *f is holomorphic;*
2. *the composite*

$$X \xrightarrow{f} H_{\mathbb{Z}} \backslash H_{\mathbb{C}} \rightarrow \mathbb{C}^* \times \mathbb{C}^*$$

of f with the canonical projection is algebraic;

3. *the map $f : X \rightarrow H_{\mathbb{Z}} \backslash H_{\mathbb{C}}$ is a flat section of the bundle $H_{\mathbb{Z}} \backslash H_{\mathbb{C}}^* \rightarrow \mathbb{C}^* \times \mathbb{C}^*$.*

Proof. The first statement corresponds to the fact that the connection on the bundle $\mathcal{V} = \mathbb{V} \otimes_{\mathbb{Z}} \mathcal{O}_X$ is holomorphic. The second follows from (9.3) and the fact that if \mathbb{V} is a variation, then so are $\mathbb{V}/\mathbb{Z}(2)$ and $W_2\mathbb{V}$. The last condition corresponds to Griffiths' transversality. One needs to use the fact that the canonical extension of \mathbb{V} to a good compactification \overline{X} of X is trivial, and that the extended Hodge and weight bundles are also trivial (8.5). \square

This result allows us to give an interpretation of the regulator

$$c_2 : K_2(X) \rightarrow H_{\mathcal{D}}^2(X, \mathbb{Z}(2))$$

constructed in Section 6: If f, g are invertible functions on X , then $c_2(\{f, g\})$ is the obstruction to finding a good variation of mixed Hodge structure \mathbb{V} over X with weight graded quotients $\mathbb{Z}, \mathbb{Z}(1), \mathbb{Z}(2)$ and whose subquotients $\mathbb{V}/\mathbb{Z}(2)$ and $W_{-2}\mathbb{V}$ are classified by

$$f : X \rightarrow \mathbb{C}^* \approx \text{Ext}_{\mathcal{H}}^1(\mathbb{Z}, \mathbb{Z}(1)) \text{ and } g : X \rightarrow \mathbb{C}^* \approx \text{Ext}_{\mathcal{H}}^1(\mathbb{Z}(1), \mathbb{Z}(2)).$$

More on extensions of variations of mixed Hodge structure can be found in [8] and [22].

10 The polylogarithm quotient of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\})$

The polylogarithm quotient of the the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ is the image of the monodromy representation

$$\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, x) \rightarrow \text{Aut } P_x$$

where $P \rightarrow \mathbb{P}^1 - \{0, 1, \infty\}$ is the polylogarithm variation of mixed Hodge structure.

Since the monodromy representation of the n th polylogarithm local system is unipotent, it induces a representation

$$\mathbb{Q}\pi_1(\mathbb{C} - \{0, 1\}, x)/J^{n+1} \rightarrow gl_{n+1}(\mathbb{C}).$$

Denote the image of the composite

$$\mathfrak{g}(X, x) \rightarrow \mathbb{Q}\pi_1(\mathbb{C} - \{0, 1\}, x)/J^{n+1} \rightarrow gl_{n+1}(\mathbb{C})$$

by $\mathfrak{p}_n(x)$.

Proposition 10.1 *For each $x \in \mathbb{C} - \{0, 1\}$, $\mathfrak{p}_n(x)$ has a natural mixed Hodge structure compatible with its Lie algebra structure. The local system of the $\mathfrak{p}_n(x)$ forms a good unipotent variation of mixed Hodge structure over $\mathbb{C} - \{0, 1\}$. Finally, these local systems form an inverse system of variations of mixed Hodge structure.*

Proof. The first assertion is an immediate consequence of (7.1) and (8.3). The second is a consequence of (8.6). The last assertion is clear. \square

By the construction given in §8, the Malcev Lie algebra of $\pi_1(\mathbb{C} - \{0, 1\}, x)$ is the completion of the free Lie algebra generated by $H_1(\mathbb{C} - \{0, 1\}, \mathbb{C}) \approx \Omega^1(\mathbb{C} - \{0, 1\})^*$. Let X_0, X_1 be the basis of $H_1(\mathbb{C} - \{0, 1\}, \mathbb{Z})$ consisting of the homology classes of the loops σ_0, σ_1 defined in §2. This is dual to the basis $\omega_0/2\pi i, -\omega_1/2\pi i$ of $\Omega^1(\mathbb{C} - \{0, 1\})$, where ω_0, ω_1 are the forms defined in §2. The completed group ring $\mathbb{C}\pi_1(\mathbb{C} - \{0, 1\}, x)^\wedge$ is isomorphic to the completion $\mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$ of the free associative algebra generated by X_0, X_1 . The set of primitive elements of $\mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$ is $\mathfrak{f} = \mathbb{L}(X_0, X_1)^\wedge$, the completion of the free Lie algebra generated by X_0 and X_1 . The isomorphism

$$\mathbb{C}\pi_1(\mathbb{C} - \{0, 1\}, x)^\wedge \rightarrow \mathbb{C}\langle\langle X_0, X_1 \rangle\rangle$$

is induced by the map $\pi_1(\mathbb{C} - \{0, 1\}, x)$ which takes γ to

$$1 + \int_{\gamma} \omega + \int_{\gamma} \omega\omega + \int_{\gamma} \omega\omega\omega + \dots$$

where ω is the \mathfrak{f} -valued 1-form

$$\omega = \omega_0 X_0 - \omega_1 X_1. \quad (15)$$

Proposition 10.2 *The monodromy representation*

$$\pi_1(\mathbb{C} - \{0, 1\}, x) \rightarrow gl_{n+1}(\mathbb{C})$$

of the polylogarithm local system is induced by the homomorphism $\mathfrak{f} \rightarrow gl_{n+1}$ defined by

$$X_0 \mapsto \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ 0 & & \cdots & & 0 \end{pmatrix} \quad X_1 \mapsto \left(\begin{array}{c|cccc} 0 & -1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 \end{array} \right)$$

Proof. This follows as the connection matrix of the polylogarithm local system is

$$\begin{pmatrix} 0 & \omega_1 & 0 & \cdots & 0 \\ & \ddots & \omega_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ & & & \ddots & \omega_0 \\ 0 & & \cdots & & 0 \end{pmatrix}$$

which is simply the \mathfrak{f} -valued form ω (15) composed the the homomorphism $\mathfrak{f} \rightarrow gl_{n+1}(\mathbb{C})$ defined in the statement of the proposition. \square

Corollary 10.3 *The complex form of the polylogarithm quotient has presentation*

$$\mathfrak{p} = \mathbb{L}(X_0, X_1)^{\sim} / (\text{ad}(X_1)[\mathbb{L}, \mathbb{L}])$$

and a topological basis $\{\text{ad}(X_0)^n(X_1) : n \in \mathbb{N}\}$. \square

11 Motivic Description of the Polylogarithm Variation

In this section we give a motivic description of the polylogarithm variations. This description goes back to Deligne.

First suppose that X is a topological space. Denote the space of paths $\gamma : [0, 1] \rightarrow X$ by PX . There is a canonical projection $PX \rightarrow X \times X$ which takes a path γ to its endpoints $(\gamma(0), \gamma(1))$. Denote the fiber of this map over (x, y) by $P_{x,y}X$.

The group $H_0(P_{x,y}X)$ is the free abelian group on the set of homotopy classes of paths in X from x to y . These form a local system

$$\{H_0(P_{x,y}X)\}_{(x,y)} \rightarrow X \times X. \quad (16)$$

When $x = y$ there is a canonical isomorphism $H_0(P_{x,y}X; \mathbb{Q}) \approx \mathbb{Q}\pi_1(X, x)$. This has a canonical filtration given by the powers of the augmentation ideal J . This filtration extends to a flat filtration of the local system (16). Denote the completion of $H_0(P_{x,y}X; \mathbb{Q})$ in the corresponding topology by $H_0(P_{x,y}X; \mathbb{Q})^\wedge$.

Theorem 11.1 [24] *If X is a smooth algebraic variety, the local system*

$$\{H_0(P_{x,y}X; \mathbb{Q})^\wedge\}_{(x,y)} \rightarrow X \times X$$

is a good variation of mixed Hodge structure whose fiber over (x, x) is the canonical mixed Hodge structure on $\mathbb{Q}\pi_1(X, x)^\wedge$.

We call this the *canonical variation of mixed Hodge structure* associated to X . Although this construction appears to be outside the domain of algebraic geometry, it can be made motivic. There are several equivalent ways of doing this. One can be found in [14]; the other is a construction in topology of a cosimplicial model of PX and the fibration $PX \rightarrow X \times X$. It is called the *cobar construction* and dates back to the paper [1] of F. Adams. It makes sense for varieties over any base, as was noted by Wojtkowiak [39]. Briefly, it is the cosimplicial space

$$P^\bullet = X^{\Delta[1]\bullet}$$

where $\Delta[1]_\bullet$ is the standard simplicial model of the unit interval. The projection $PX \rightarrow X \times X$ corresponds to the map $X^{\Delta[1]_\bullet} \rightarrow X^{\partial\Delta[1]_\bullet} = X^2$ induced by the inclusion of the boundary of the interval.

Since the set of m -simplices of $\Delta[1]_\bullet$ is the set of order preserving maps $\{0, 1, \dots, m\} \rightarrow \{0, 1\}$, and since there are precisely $m + 2$ of these,

$$P^m = X \times X^m \times X.$$

The coface maps $P^m \rightarrow P^{m+1}$ are the various diagonals. Applying the de Rham functor to this cosimplicial space yields the bar construction on the de Rham complex of X , which is essentially Chen's complex of iterated integrals on PX ([9], see also [21, §§1,2]).

Suppose that $X = \overline{X} - D$ is a smooth curve and that \vec{v} is a tangent vector at a point $P \in D$. Suppose that $x \in X$. Denote by $P_{\vec{v},x}X$ the set of paths $\gamma : [0, 1] \rightarrow \overline{X}$ which have the property that $\gamma'(0) = \vec{v}$, $\gamma(1) = x$ and $\gamma([0, 1]) \subseteq X$. This space is easily seen to be homotopy equivalent to $P_{z,x}X$ where z is a point in X which is sufficiently close to P in the direction of \vec{v} . More generally, one can define $P_{\vec{v}_1, \vec{v}_2}X$ where \vec{v}_1 and \vec{v}_2 are non-zero tangent vectors to points of D . Deligne defines $\pi_1(X, \vec{v})$ to be the set of path components of $P_{\vec{v}, \vec{v}}X$. It is canonically isomorphic to $\pi_1(X, z)$ when $z \in X$ is sufficiently close to P in the direction of \vec{v} (cf. [14]).

It is useful to think of the limit mixed Hodge structure of the local system

$$\{H_0(P_{z,x}X; \mathbb{Q})^\wedge\}_{z \in X} \rightarrow X$$

associated to \vec{v} as a mixed Hodge structure on $H_0(P_{\vec{v},x}X; \mathbb{Q})^\wedge$. Denote the limit mixed Hodge structure on the Malcev Lie algebra of the polylogarithm quotient of $\pi_1(\mathbb{C} - \{0, 1\}, \vec{v})$ associated to the tangent vector \vec{v} by $\mathfrak{p}(\vec{v})$.

Theorem 11.2 *Let \vec{v} be the tangent vector $\partial/\partial z$ at $0 \in \mathbb{P}^1$. The polylogarithm local system is the quotient of the variation of mixed Hodge structure*

$$\{H_0(P_{\vec{v},z}\mathbb{C} - \{0, 1\}; \mathbb{Q})^\wedge\}_{z \in \mathbb{C} - \{0, 1\}} \rightarrow \mathbb{C} - \{0, 1\}$$

whose fiber over \vec{v} is the polylogarithm quotient $\mathfrak{p}(\vec{v})$ of $\pi_1(\mathbb{C} - \{0, 1\}, \vec{v})$.

Since both variations are isomorphic as local systems, to prove the theorem it suffices to show that the fibers of the two variations over one

particular point (or tangent vector) are isomorphic as mixed Hodge structures. It is not difficult to show that the fiber of the quotient of the canonical variation over the tangent vector $\partial/\partial z$ at 0 is isomorphic to that of the polylog variation, which was calculated in (7.2).

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